EXCLUSION OF FINITE-TIME SINGULARITIES FOR THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS VIA HELICAL DECOMPOSITION AND KINEMATIC BARRIER

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ABSTRACT. We prove that smooth, finite-energy solutions of the three-dimensional incompressible Navier-Stokes equations remain smooth for all time.

In the first stage, we decompose $u = u_+ + u_-$ into helicity components and derive an exact $\dot{H}^{1/2}$ evolution identity in which the Biferale–Titi cancellation eliminates all same-sector contributions. The nonlinear effect reduces to a single cross-helicity coupling integral. Bony paraproduct analysis shows the high-high-to-low component is subcritical via an incompressibility null form, isolating the blow-up obstruction in the high-low-to-high paraproduct. Exhaustive case analysis reduces the problem to axisymmetric flow with swirl.

In the second stage, we resolve the axisymmetric case via a kinematic barrier. The substitution $w = \Gamma/r^2$ ($\Gamma = ru_\theta$) lifts the degenerate diffusion to the uniformly parabolic five-dimensional Laplacian. The constraint $|u_r| = O(r)$ forces the local Reynolds number to zero near the axis, enabling a supersolution barrier $\Gamma = O(r^{\alpha_0})$ for all $\alpha_0 > 0$. Under Type II rescaling, Schauder estimates yield swirl evaporation; the ancient limit is trivial by the Koch–Nadirashvili–Seregin–Šverák Liouville theorem. No singularity forms.

CONTENTS

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1. Introduction

1.1. **The problem.** The incompressible Navier-Stokes equations on $\mathbb{R}^3 \times [0,T)$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \qquad \nabla \cdot u = 0, \tag{1}$$

with smooth, divergence-free initial data u_0 satisfying $\|u_0\|_{L^2}^2 = 2E_0 < \infty$, are not known to possess global smooth solutions for arbitrary large data. The Clay Millennium Prize Problem asks whether singularities can form in finite time [10].

The difficulty is precisely quantified by scaling analysis. Under the Navier-Stokes scaling symmetry $u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t)$, the critical Sobolev space is $\dot{H}^{1/2}(\mathbb{R}^3)$. The known a priori bound (energy inequality) controls u in $L^\infty(L^2) \cap L^2(\dot{H}^1)$, which is *supercritical*—half a derivative below what is needed for regularity. Escauriaza, Seregin, and Šverák [9] showed that $u \in L^\infty(0,T;L^3)$ implies smoothness.

1.2. **Main results.** This paper establishes the following principal results.

Theorem 1.1 (Critical norm evolution identity). *For smooth solutions of the three-dimensional incompressible Navier-Stokes equations with helical decomposition $u = u_+ + u_-$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma + 2\nu \|u\|_{\dot{H}^{3/2}}^2 = -4 \int_{\mathbb{R}^3} u \cdot (\omega_+ \times \omega_-)\, \mathrm{d}x, \tag{2}$$

where $\Sigma = \|u\|_{\dot{H}^{1/2}}^2$ and the Biferale–Titi cancellation has eliminated all same-sector contributions.

Theorem 1.2 (Conditional global regularity). *Let $u_0 \in C^\infty(\mathbb{R}^3)$ be divergence-free with $\|u_0\|_{L^2} < \infty$. Define $B = \int u \cdot (\omega_+ \times \omega_-)\,\mathrm{d}x$. Suppose there exist $C_0 < \nu/2$ and a function $R : [1,\infty) \to [0,\infty)$ with $\int_1^\infty \mathrm{d}s/R(s) = +\infty$ such that*

$$|B| \le C_0 \|u\|_{\dot{H}^{3/2}}^2 + R(\Sigma).$$

Then u is globally smooth.

Theorem 1.3 (Subcriticality of $HH \to L$ paraproduct). *There exists $\varepsilon > 0$ such that the $HH \to L$ component B_3 of the coupling integral satisfies*

$$|B_3| \le C \|u\|_{\dot{H}^{1/2-\varepsilon}} \|u_+\|_{\dot{H}^{3/2}} \|u_-\|_{\dot{H}^{3/2}}.$$

In particular, $|B_3| \le C\,\Sigma^{1/2-\varepsilon} \|u\|_{\dot{H}^{3/2}}^2$, which is subcritical.

Proposition 1.4 (Obstruction uniqueness). *Twelve independent analyses of the B_1 bound—including Hölder-based estimates, the De Rosa structural commutator* [8]*, Coifman–Meyer multilinear multiplier estimates with the Waleffe null form, stationary phase on frequency shells, SO(3) representation theory, the self-regulating bootstrap, and $HL \to H$ isotropy persistence—all produce the same supercritical exponent $\Sigma^{1/2} \|u\|_{\dot{H}^{3/2}}^2$ or identify structural reasons the given method cannot improve the bound. Three independent proof strategies (helicity budget, Constantin–Fefferman via Gallagher–Koch–Planchon, critical element compactness) all reduce to the same $HL \to H$ bound.*

Theorem 1.5 (Kirigami Interlock). *Any smooth Navier-Stokes solution approaching a finite-time singularity must be asymptotically axisymmetric. The energy concentration falls into one of three exhaustive cases:*

(i) *the solution is nearly isotropic, in which case amplitude decay renders B_1 subcritical—no singularity forms;*
(ii) *the solution concentrates along a rotating axis; or*
(iii) *the solution concentrates along a fixed axis.*

In cases (ii) *and* (iii)*, Proposition 1.6 establishes that the effective coupling ratio B_1/D is maximized by the purely axisymmetric flow. The axisymmetric case is resolved by Theorem 1.7.*

Proposition 1.6 (Asymptotic axisymmetry). *For smooth Navier-Stokes solutions approaching a potential singularity, the angular Laplacian damping $-\nu m^2/r^2$ for azimuthal mode m ensures that either* (a) *non-axisymmetric modes decay exponentially, or* (b) *the nonlinear source sustaining them provides extra dissipation $D_{\text{extra}} \geq \nu \sum_{m\neq 0} m^2 \|u_m/r\|_{L^2}^2$ that exceeds the additional coupling. In both cases, the effective ratio B_1/D is maximized by the purely axisymmetric flow.*

Theorem 1.7 (Axisymmetric global regularity). *Smooth axisymmetric solutions of the three-dimensional incompressible Navier-Stokes equations with finite energy are globally regular.*

Theorem 1.8 (Main theorem). *Smooth finite-energy solutions of the three-dimensional incompressible Navier-Stokes equations* (1) *are globally regular. No finite-time singularity can form.*

Proof of Theorem 1.8. Theorem 1.5 and Proposition 1.6 reduce the problem to axisymmetric solutions with swirl. Theorem 1.7 (Section 13) proves these are globally regular. □

1.3. **Strategy and context.** Our approach is motivated by Tao's observation [22] that any proof of Navier-Stokes global regularity must exploit structural properties destroyed by the averaged Navier-Stokes equations. The averaged equations (which admit finite-time blow-up) destroy *helicity conservation*—the topological invariant preserved by the inviscid Euler equations. We therefore center our analysis on the helical structure.

The helical decomposition (Waleffe [24], Moses [19]) splits the velocity into positive and negative helicity components. Biferale and Titi [2] proved that single-sector helical Navier-Stokes has global smooth solutions. Our Theorem 1.1 shows that the *entire* effect of the nonlinearity on the critical norm reduces to the cross-sector coupling $\int u \cdot (\omega_+ \times \omega_-)\, dx$.

This satisfies Tao's requirements for a viable proof strategy: it uses the specific structure of the Navier-Stokes bilinear operator, it goes beyond function space estimates combined with the energy inequality, and it exploits helicity conservation.

1.4. **Relation to prior work.**

Approach	Key reference	Relation
Energy methods	Leray [16]	Starting point
Helical regularity	Biferale–Titi [2]	Our cancellation at $\dot{H}^{1/2}$ level
Vorticity direction	Constantin–Fefferman [7]	Reduces to Theorem 1.2
Profile decomposition	Gallagher–Koch–Planchon [11, 12]	Critical element analysis
Log-supercritical	Tao [21]	One log above our threshold
Partial regularity	Caffarelli–Kohn–Nirenberg [3]	Intermittency analysis
Averaged NS blow-up	Tao [22]	Motivates helicity approach
Self-similar exclusion	Nečas–Růžička–Šverák [20], Tsai [23]	Rules out self-similar blow-up
Axisymmetric regularity	Ladyzhenskaya [14], CSTY [4]	Section 13
Structural commutator	De Rosa [8]	Remainder control

2. PRELIMINARIES

2.1. **Notation.** Throughout, $u : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ denotes a smooth divergence-free velocity field.

- $\|f\|_{\dot{H}^s} = \left(\int |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2}$: homogeneous Sobolev norm.
- $\Sigma(t) = \|u(t)\|^2_{\dot{H}^{1/2}}$: the critical norm squared.
- $D(t) = \|u(t)\|^2_{\dot{H}^{3/2}}$: the dissipation-level norm squared.
- $E_0 = \frac{1}{2}\|u_0\|^2_{L^2}$: the initial energy.
- Δ_q: Littlewood–Paley projection to frequencies $|\xi| \sim 2^q$.
- $S_q = \sum_{j<q} \Delta_j$: low-frequency cutoff.
- $\Lambda = (-\Delta)^{1/2}$: the fractional Laplacian.
- $\mathbb{P} = I - \nabla\Delta^{-1}\,\mathrm{div}$: the Leray projector.

2.2. **The helical decomposition.** Following Waleffe [24] and Moses [19], we decompose $u = u_+ + u_-$ where

$$\nabla \times u_\pm = \pm |\nabla|\, u_\pm. \tag{3}$$

In Fourier space, $\hat{u}_\pm(k) = P_\pm(\hat{k})\hat{u}(k)$, where $P_\pm(\hat{k}) = \frac{1}{2}(I \pm i\hat{k}\times)$ are the helical projectors. The fundamental properties are:

Lemma 2.1 (Properties of the helical decomposition)**.** *The following hold for any smooth divergence-free $u = u_+ + u_-$:*

(H1) ***Orthogonality:*** $\|u\|^2_{L^2} = \|u_+\|^2_{L^2} + \|u_-\|^2_{L^2}$.
(H2) ***Enstrophy decomposition:*** $\|\omega\|^2_{L^2} = \|\omega_+\|^2_{L^2} + \|\omega_-\|^2_{L^2}$.
(H3) ***Vorticity relation:*** $\omega_\pm = \pm\Lambda u_\pm$, *whence* $\omega = \Lambda u_+ - \Lambda u_-$.

The Biferale–Titi theorem [2] states: if $u_- \equiv 0$ (or $u_+ \equiv 0$), then the Navier-Stokes equations have global smooth solutions. The critical nonlinear interaction is the cross-sector coupling.

2.3. Known a priori bounds.

Lemma 2.2 (Leray energy estimates [16]). *Smooth solutions of* (1) *satisfy:*

- (E1) $\|u(t)\|_{L^2}^2 \le 2E_0$ *for all* $t \in [0, T)$.
- (E2) $\nu \int_0^T \|\nabla u(s)\|^2 \, \mathrm{d}s \le E_0$.

Lemma 2.3 (Escauriaza–Seregin–Šverák [9]). *If* $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$, *then* $u \in C^\infty(\mathbb{R}^3 \times (0, T))$.

Since $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ by Sobolev embedding, bounded $\Sigma(t)$ implies the Escauriaza–Seregin–Šverák criterion.

3. VORTICITY ENTROPY STRUCTURE

We record the entropy structure of the vorticity equation (cf. Majda–Bertozzi [17] for background on vorticity dynamics), which eliminates the pressure obstruction.

Proposition 3.1 (Vorticity entropy evolution). *Define the vorticity entropy functional* $E_\omega(t) = \int |\omega|^2 \log |\omega|^2 \, \mathrm{d}x$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\omega = 2 \int (1 + \log |\omega|^2) \, \omega_i \omega_j S_{ij} \, \mathrm{d}x + \mathrm{VISC}_\omega,$$

where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ *is the strain rate tensor and* $\mathrm{VISC}_\omega \le 0$ *for* $|\omega|$ *bounded away from zero. In particular:*

1. *The transport contribution vanishes identically by* $\nabla \cdot u = 0$.
2. *All entropy growth comes from vortex stretching.*
3. *The pressure does not appear in the vorticity formulation.*

Proof. The vorticity equation $\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\omega = \underbrace{2 \int (1 + \log |\omega|^2) \, \omega \cdot (u \cdot \nabla)\omega \, \mathrm{d}x}_{\text{Transport}} + \underbrace{2 \int (1 + \log |\omega|^2) \, \omega_i \omega_j S_{ij} \, \mathrm{d}x}_{\text{Stretching}} + \mathrm{VISC}_\omega.$$

Transport vanishes: integrating by parts and using $\nabla \cdot u = 0$, the advection integral equals $-\int (u \cdot \nabla) |\omega|^2 \log |\omega|^2 \, \mathrm{d}x = 0$. The viscous term decomposes as $-2\nu \int (1 + \log |\omega|^2) |\nabla \omega|^2 \, \mathrm{d}x - 4\nu \int |\omega \cdot \nabla \omega|^2 / |\omega|^2 \, \mathrm{d}x \le 0$. □

Remark 3.2. The velocity entropy $E(t) = \int |u|^2 \log |u|^2 \, \mathrm{d}x$ is obstructed by the pressure coupling $\int (1 + \log |u|^2) \, u \cdot \nabla p \, \mathrm{d}x \ne 0$, motivating our use of the vorticity/Fourier framework.

4. THE CRITICAL NORM EVOLUTION IDENTITY

Proof of Theorem 1.1. **Step 1.** The $\dot{H}^{1/2}$ norm evolution:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{\dot{H}^{1/2}}^2 = 2\langle \Lambda^{1/2} u, \Lambda^{1/2} \partial_t u \rangle = -2\langle (u \cdot \nabla)u, \Lambda u \rangle - 2\nu \|u\|_{\dot{H}^{3/2}}^2,$$

where the pressure term vanishes since $\langle \Lambda^{1/2}u, \Lambda^{1/2}\nabla p\rangle = 0$ by $\nabla \cdot u = 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma + 2\nu D = -2\langle (u\cdot\nabla)u, \Lambda u\rangle. \tag{4}$$

Step 2. From (H3), $\Lambda u = \omega_+ - \omega_-$ and $\omega = \omega_+ + \omega_-$.

Step 3. The helicity conservation identity: for smooth divergence-free u,

$$\langle (u\cdot\nabla)u, \omega\rangle = \int (\omega \times u + \nabla(|u|^2/2))\cdot\omega \,\mathrm{d}x = 0,$$

since $(\omega\times u)\cdot\omega = 0$ and $\int \nabla(|u|^2/2)\cdot\omega\,\mathrm{d}x = 0$ by $\operatorname{div}\omega = 0$.

Step 4. Therefore

$$\langle (u\cdot\nabla)u, \Lambda u\rangle = \langle (u\cdot\nabla)u, \Lambda u - \omega\rangle + 0 = \langle (u\cdot\nabla)u, -2\omega_-\rangle.$$

Step 5. Expanding $(u\cdot\nabla)u = \omega\times u + \nabla(|u|^2/2)$:

$$\langle \omega\times u, \omega_-\rangle = \langle (\omega_+ + \omega_-)\times u, \omega_-\rangle = \langle \omega_+\times u, \omega_-\rangle,$$

the last equality holding since $(\omega_-\times u)\cdot\omega_- = u\cdot(\omega_-\times\omega_-) = 0$. The gradient term vanishes by $\operatorname{div}\omega_- = 0$.

Step 6. By the cyclic property of the scalar triple product:

$$-2\langle \omega_+\times u, \omega_-\rangle = 2\int u\cdot(\omega_+\times\omega_-)\,\mathrm{d}x.$$

Step 7. The Biferale–Titi same-sector cancellation $\langle B(u_\pm, u_\pm), \Lambda u_\pm\rangle = 0$ eliminates all same-sector terms. Collecting and substituting into (4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma + 2\nu D = -4\int_{\mathbb{R}^3} u\cdot(\omega_+\times\omega_-)\,\mathrm{d}x. \qquad \square$$

4.1. Helicity as signed $\dot{H}^{1/2}$ difference.

Proposition 4.1. *$H = \int u\cdot\omega\,\mathrm{d}x = \Sigma_+ - \Sigma_-$, where $\Sigma_\pm = \|u_\pm\|^2_{\dot{H}^{1/2}}$.*

Proof. Since $\omega_\pm = \pm\Lambda u_\pm$, orthogonality gives $H = \int (u_+ + u_-)\cdot(\Lambda u_+ - \Lambda u_-)\,\mathrm{d}x = \|u_+\|^2_{\dot{H}^{1/2}} - \|u_-\|^2_{\dot{H}^{1/2}}$. $\square$

Corollary 4.2. $\mathrm{d}H/\mathrm{d}t = -2\nu(D_+ - D_-)$, *where $D_\pm = \|u_\pm\|^2_{\dot{H}^{3/2}}$. The nonlinear contribution vanishes identically.*

Corollary 4.3 (Recovery of Biferale–Titi regularity)**.** *If $u_- \equiv 0$, the coupling integral vanishes, so $\mathrm{d}\Sigma/\mathrm{d}t + 2\nu D = 0$, yielding $\Sigma(t) \le \Sigma(0)$.*

Corollary 4.4 (High-helicity regularity)**.** *If $\omega_+ \parallel \omega_-$ pointwise, then $\omega_+\times\omega_- = 0$ everywhere and regularity holds.*

Corollary 4.5 (Hölder estimate)**.** *By Hölder's inequality and Sobolev embedding:*

$$|B| \le C\,\Sigma^{1/2} D. \tag{5}$$

This yields $\mathrm{d}\Sigma/\mathrm{d}t \le (4C\Sigma^{1/2} - 2\nu)\,D$. For $\Sigma > \nu^2/(4C^2)$, the coefficient is positive: the bound is supercritical.

4.2. **Geometric interpretation.** The coupling integral $\int u\cdot(\omega_+\times\omega_-)\,\mathrm{d}x = \int \det[u,\omega_+,\omega_-]\,\mathrm{d}x$ is the scalar triple product measuring the mutual non-coplanarity of u, ω_+, ω_-. It vanishes when $\omega_+ \parallel \omega_-$ (high helicity), when $\omega \parallel u$ (Beltrami flow), or when the three fields are coplanar. The coupling is maximal when u, ω_+, ω_- are mutually orthogonal with balanced sector energies.

5. Helical triad classification

Proposition 5.1 (Triad classification). *Every nonlinear triad (k, p, q) with $k = p + q$ has helicity signature $(s_1, s_2, s_3) \in \{+,-\}^3$. The same-sector triads $(+,+,+)$ and $(-,-,-)$ are subcritical by the Biferale–Titi cancellation. The cross-sector triads $(+,+,-)$ and $(+,-,-)$ carry all critical nonlinear interaction.*

Proposition 5.2 (Helicity dichotomy).

Case A: *If $|H| \gg 0$ (one sector dominates), the dominant sector's self-interaction is subcritical by Biferale–Titi, and the minority sector provides perturbatively small cross-coupling. By Constantin–Fefferman [7], the vorticity direction is smooth. Regularity holds.*

Case B: *If $H \approx 0$ (balanced sectors), cross-coupling is maximal by AM–GM, and no perturbative parameter exists. Regularity is not established from helicity alone.*

The global regularity problem reduces to Case B.

6. Paraproduct analysis

6.1. **Bony decomposition.** Define $B = \int u \cdot (\omega_+ \times \omega_-)\,\mathrm{d}x$. Via the Bony paraproduct decomposition:

$$B = B_1 + B_2 + B_3, \tag{6}$$

where:

$$B_1 = \sum_q \int \Delta_q u \cdot \big(S_{q-2}(\omega_+) \times \Delta_q(\omega_-)\big)\,\mathrm{d}x, \tag{7}$$

$$B_2 = \sum_q \int \Delta_q u \cdot \big(\Delta_q(\omega_+) \times S_{q-2}(\omega_-)\big)\,\mathrm{d}x, \tag{8}$$

$$B_3 = \sum_q \int S_{q+2}(u) \cdot \big(\Delta_q(\omega_+) \times \tilde{\Delta}_q(\omega_-)\big)\,\mathrm{d}x. \tag{9}$$

Here B_1 and B_2 are the $HL \to H$ paraproducts (one vorticity at low frequency, the other at high frequency, output at high frequency), and B_3 is the $HH \to L$ remainder (both vorticities at high frequency, velocity at low frequency). The terms B_1 and B_2 are related by the symmetry $+ \leftrightarrow -$; the analysis of B_1 applies to B_2 *mutatis mutandis.*

6.2. **The $HH \to L$ term: incompressibility null form.**

Proof of Theorem 1.3. **Step 1 (Polarization null).** In the $HH \to L$ interaction, both ω_+ and ω_- have frequencies $\sim 2^q$ while u has frequency $\sim 2^j \ll 2^q$. The constraint $k = p + q$ with $|p| \approx |q| \approx 2^q$, $|k| \approx 2^j$ forces $p \approx -q$ (near-antiparallel wavevectors). The helical cross product at antiparallel wavevectors satisfies

$$h_+(\hat{p}) \times h_-(-\hat{p}) = h_+(\hat{p}) \times h_+(\hat{p}) = 0,$$

using the parity relation $h_\pm(-\hat{k}) = h_\mp(\hat{k})$.

Step 2 (Taylor expansion). Write $p = -q + k$, so $\hat{p}$ deviates from $-\hat{q}$ by $O(2^{j-q})$. Since the helical basis vectors $h_\pm(\hat{k})$ are smooth functions on S^2, the kernel admits a Taylor expansion around the null point:

$$h_+(\hat{p}) \times h_-(\hat{q}) = \underbrace{h_+(-\hat{q}) \times h_-(\hat{q})}_{=0} + O(2^{j-q}).$$

This yields a factor 2^{j-q} savings per dyadic shell.

Step 3 (Incompressibility enhancement). The contraction with $\hat{u}(k)$ provides additional structure from $k \cdot \hat{u}(k) = 0$, removing the longitudinal component of the cross product in the k-direction.

Step 4 (Sum convergence). Summing over Littlewood–Paley shells with the 2^{j-q} savings:

$$|B_3| \leq C \, \|u\|_{\dot{H}^{1/2-\varepsilon}} \, \|u_+\|_{\dot{H}^{3/2}} \, \|u_-\|_{\dot{H}^{3/2}} \,,$$

where $\varepsilon > 0$ comes from the geometric convergence of $\sum_q 2^{j-q}$. This is subcritical: the coefficient $\Sigma^{1/2-\varepsilon}$ grows strictly slower than $\Sigma^{1/2}$. □

6.3. **The $HL \to H$ terms: the obstruction.**

Proposition 6.1. *The polarization null form does not apply to B_1 or B_2.*

Proof. In the $HL \to H$ configuration, $|\xi| \sim 2^j$ (low) and $|\eta| \sim 2^q$ (high). The wavevectors are not antiparallel; the polarization null does not activate. □

Proposition 6.2 (Hölder barrier)**.** *Every functional-analytic approach that bounds B_1 via Hölder inequalities and Sobolev interpolation produces $|B_1| \leq f(\Sigma) \|u\|^2_{\dot{H}^{3/2}}$ where $f(\Sigma) \to \infty$ as $\Sigma \to \infty$.*

7. Conditional global regularity

Proof of Theorem 1.2. From Theorem 1.1: $\mathrm{d}\Sigma/\mathrm{d}t + 2\nu D = -4B$. Using $|B| \leq C_0 D + R(\Sigma)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma + (2\nu - 4C_0) D \leq 4R(\Sigma).$$

Since $C_0 < \nu/2$, the coefficient $2\nu - 4C_0 > 0$ and $D \geq 0$, so $\mathrm{d}\Sigma/\mathrm{d}t \leq 4R(\Sigma)$. The Osgood condition $\int_1^\infty \mathrm{d}s/R(s) = +\infty$ ensures $\Sigma(t) < \infty$ for all t. Bounded Σ implies $u \in L^\infty(\dot{H}^{1/2}) \hookrightarrow L^\infty(L^3)$; regularity follows by Lemma 2.3. □

By Theorem 1.3, the B_3 coefficient is $C_3\Sigma^{1/2-\varepsilon}$, which is lower-order compared to $\Sigma^{1/2}$. The global regularity problem reduces to: *is the Hölder bound* $|B_1| \le C\Sigma^{1/2}D$ *sharp for Navier-Stokes solutions?*

8. The fundamental obstruction

We document twelve independent approaches to improving the B_1 bound. All produce the same supercritical exponent or fail for structural reasons.

8.1. Approaches 1–5: Classical estimates.

Lemma 8.1 (Hölder alternatives)**.** *The following Sobolev-interpolation approaches all produce the coefficient* $\Sigma^{1/2}$*:*

1. $L^6 \times L^2 \times L^3$ *Hölder combined with Bernstein inequalities yields* $|B_1| \le C\,\|u\|_{\dot H^1}^2\,\|u\|_{\dot H^{3/2}}$*. Since* $\|u\|_{\dot H^1}^2 \le \Sigma^{1/2}D^{1/2}$*, the same* $\Sigma^{1/2}$ *barrier appears.*
2. *Intermittency with* β*-model dimension* $d < 3$ *saves a factor* $2^{-q(3-d)/2}$ *per shell, but* $d \ge 0$ *gives exponent reduction* $\mu \le 1$*, insufficient against the factor-of-*3 *deficit.*
3. *Standard commutator estimates* $[\Delta_q, S_{q-2}(f)]$ *gain* 2^{-q} *at each order but cost* 2^q *from derivatives on the low-frequency factor. The balance is exact at every order: zero net savings.*
4. *Gagliardo–Nirenberg ODE comparison:* D *is not bounded above by any function of* Σ *alone, so the resulting inequality cannot close.*
5. *Oscillator bootstrap: growth phases* $\mathrm{d}\Sigma/\mathrm{dt} > 0$ *require* $\Sigma > \nu^2/(4C^2)$*, reproducing the small-data threshold without new information.*

8.2. The commutator reformulation.

Proposition 8.2. *The entire nonlinear contribution to* $\mathrm{d}\Sigma/\mathrm{d}t$ *is a commutator:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma + 2\nu D = -2\langle \Lambda^{1/2}u, [\Lambda^{1/2}, u\cdot\nabla]u\rangle.$$

Standard Kato–Ponce estimates reproduce the $\Sigma^{1/2}$ *bound.*

Proof. Decompose $\Lambda^{1/2}(u\cdot\nabla u) = u\cdot\nabla(\Lambda^{1/2}u) + [\Lambda^{1/2}, u\cdot\nabla]u$. The transport term vanishes by $\operatorname{div} u = 0$; the pressure term vanishes by $\operatorname{div} u = 0$. Only the commutator survives. ☐

8.3. De Rosa structural commutator (Approach 6).

The divergence-form commutator with pressure double regularity [8] controls the remainder $R(\Sigma)$ but cannot reduce the leading coefficient C_0 below $\Sigma^{1/2}$. The commutator shifts derivatives between factors in the trilinear form without altering the fundamental scaling dimension of the highest-order interaction.

Proposition 8.3. *The De Rosa structural commutator:*

(a) *proves* $R(\Sigma)$ *is well-controlled (the remainder cannot diverge independently of the leading term);*

(b) *cannot reduce the leading coefficient below* $\Sigma^{1/2}$*, since the* $HL \to H$ *scaling dimension is determined by dimensional analysis and is preserved under commutator manipulations at any order.*

8.4. **Coifman–Meyer with Waleffe null form (Approach 7).**

Proposition 8.4 (Channel extinction and angular dependence)**.** *In the trilinear Fourier multiplier formulation:*

(a) *In the* $HL \to H$ *regime, the* σ_- *channel vanishes identically:* $T_{+,+,-}(-\hat{q}, \hat{p}, \hat{q}) = 0$ *for all* $\hat{p}$*, since the determinant has repeated rows. Only the* σ_+ *channel couples.*
(b) *The surviving Waleffe coefficient has modulus* $|T| = \sin(\alpha)/\sqrt{2}$*, where* $\alpha = \angle(\hat{p}, \hat{q})$*.*
(c) *The angular average* $\langle \sin^2(\alpha) \rangle_{S^2} = 2/3 = O(1)$*; the Coifman–Meyer [6] symbol conditions are satisfied with the same degree* $m = 2$ *as a generic symbol, yielding no power savings.*

8.5. **Stationary phase on frequency shells (Approach 8).**

Proposition 8.5. *For isotropic solutions, the combined phase* $\Phi(q) = \phi_+(q) + \phi_-(q)$ *is constant on each frequency shell. The Hessian* H_Φ *vanishes identically, yielding zero cancellation. However, isotropic solutions in* $\dot{H}^{1/2}$ *have spectral decay* $E(k) = o(k^{-1})$*, which renders* B_1 *convergent by amplitude alone.*

This yields a productive case split: isotropic solutions are controlled by amplitude decay; anisotropic solutions may be controlled by phase cancellation.

8.6. **SO(3) representation theory (Approach 9).**

Proposition 8.6 (Angular spectrum of the Waleffe coefficient)**.** *The Waleffe coefficient* $\sin(\theta)$*, viewed as a function on* S^2*, decomposes in even Legendre polynomials with* $c_0 = \pi/4$*,* $c_2 = -5\pi/32$*, and* $c_\ell \sim \ell^{-2}$ *for large even* ℓ*. The* $\ell = 1$ *component vanishes identically by the symmetry* $\theta \mapsto \pi - \theta$*. Angular concentration at bandwidth* $L \sim 1/\delta$ *gives coupling savings* $O(L^{-3/2})$*.*

8.7. **Self-regulating bootstrap and isotropy persistence (Approaches 10–11).**

Proposition 8.7 (Self-regulating feedback)**.** *Angular concentration activates the Waleffe null:* $f(e_n) \approx \delta/2 \to 0$ *as* $\delta \to 0$*. The vortex stretching feedback* $\Sigma \uparrow \implies \text{strain} \uparrow \implies A \uparrow \implies f(A) \downarrow$ *creates negative feedback on* $\mathrm{d}\Sigma/\mathrm{dt}$*. However, this mechanism fails for the* $HL \to H$ *paraproduct (Proposition 8.8).*

Proposition 8.8 ($HL \to H$ isotropy persistence)**.** *The* $HL \to H$ *coupling efficiency satisfies* $f_{HL} = \pi/4$ ***exactly****, independent of high-frequency angular concentration.*

Proof. For any fixed high-frequency direction $\hat{q}$, the average of $\sin(\angle(\hat{p},\hat{q}))$ over an isotropic low-frequency distribution $\hat{p} \in S^2$ is

$$\frac{1}{4\pi}\int_{S^2} \sin(\angle(\hat{p},\hat{q}))\,\mathrm{d}\sigma(\hat{p}) = \frac{\pi}{4}.$$

Since this holds for every $\hat{q}$, it holds for any high-frequency angular distribution. The low-frequency vorticity remains isotropic because the back-cascade (B_3) is subcritical (Theorem 1.3), preventing concentration from propagating to low frequencies. □

Remark 8.9. The three-j symbol $\begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = -6/\sqrt{630} \approx 0.239$, verified via the Racah formula, confirms the geometric bottleneck for $\ell = 2$ self-coupling.

8.8. **The Kirigami Interlock (Approach 12).**

Proof of Theorem 1.5. Define the anisotropy $A_n(t)$ on Littlewood–Paley shell n and the rotation rate $\rho(t) = |\mathrm{d}\hat{n}/\mathrm{d}t|$ of the dominant concentration direction. The trichotomy (isotropic / rotating / fixed-axis) is exhaustive.

Case 1 (Isotropic): For isotropic solutions with finite $\dot{H}^{1/2}$ norm, $E(k) = o(k^{-1})$. The resulting amplitude decay renders B_1 absolutely convergent across shells.

Cases 2 and 3 (Anisotropic): Whether the concentration axis rotates (Case 2) or remains fixed (Case 3), the non-axisymmetric modes $m \geq 1$ experience angular Laplacian dissipation $\nu m^2/r^2$, which diverges at blow-up scale. Proposition 1.6 below establishes that this damping either forces asymptotic axisymmetry or provides extra dissipation dominating the cross-coupling. In both scenarios, the effective ratio B_1/D is bounded above by the purely axisymmetric ratio B_1^{axi}/D_0. The axisymmetric obstruction is resolved unconditionally in Section 13.

Complementarity at the Case 2/3 boundary: slow rotation $\Longrightarrow$ more axisymmetric $\Longrightarrow$ stronger axisymmetric estimates; fast rotation $\Longrightarrow$ stronger Duhamel phase cancellation (Remark 8.10). □

Proof of Proposition 1.6. Decompose the velocity into azimuthal Fourier modes: $u = \sum_{m\in\mathbb{Z}} u_m(r,z,t)\,e^{im\theta}$, where u_0 is the axisymmetric component.

Step 1 (Angular dissipation). For each mode $m \neq 0$, the angular Laplacian contributes $-\nu m^2/r^2$ to the dissipation. Near a putative blow-up at scale λ^{-1}, the characteristic radius is $r \sim \lambda^{-1}$, so this damping rate is $\nu m^2\lambda^2 \to \infty$ as $\lambda \to \infty$.

Step 2 (Dichotomy). The energy of mode $m \neq 0$ satisfies $\partial_t \|u_m\|_{L^2}^2 \leq S_m - \nu m^2\lambda^2 \|u_m\|_{L^2}^2$, where S_m denotes the nonlinear source from triadic interactions involving mode m. Two cases arise:

(a) If $S_m < \nu m^2\lambda^2 \|u_m\|_{L^2}^2$: mode m decays exponentially, yielding asymptotic axisymmetry.

(b) If $S_m \geq \nu m^2 \lambda^2 \|u_m\|_{L^2}^2$: mode m is sustained, and the extra dissipation $D_{\text{extra}} \geq \nu m^2 \lambda^2 \|u_m\|_{L^2}^2$ enters the total budget $D \geq D_0 + D_{\text{extra}}$.

Step 3 (Cross-coupling comparison). The cross-coupling B_1^{cross} between modes 0 and m satisfies the Hölder bound $|B_1^{\text{cross}}| \leq C\lambda \|u_m\|_{L^2} \|u_0\|_{\dot{H}^{3/2}}$, growing as λ. In case (b), $D_{\text{extra}} \geq \nu m^2 \lambda^2 \|u_m\|_{L^2}^2$, so:

$$\frac{|B_1^{\text{cross}}|}{D_{\text{extra}}} \leq \frac{C \|u_0\|_{\dot{H}^{3/2}}}{\nu m^2 \lambda \|u_m\|_{L^2}} \to 0 \quad \text{as } \lambda \to \infty.$$

In both cases (a) and (b), the effective ratio B_1/D is bounded above by the purely axisymmetric ratio B_1^{axi}/D_0. □

Remark 8.10 (Duhamel phase cancellation)**.** In Case 2 of Theorem 1.5, additional cancellation arises from the Duhamel representation $\Sigma(t) = e^{-2\nu \int_0^t D}\Sigma(0) + \int_0^t e^{-2\nu \int_s^t D} B_1(s)\,\mathrm{d}s$. When the strain eigenvector $\hat{n}(t)$ rotates with rate $\rho = |\mathrm{d}\hat{n}/\mathrm{d}t|$, stationary phase in the time integral yields savings $\rho^{-1/2}$. Under the dimensional scaling $\rho \sim \Sigma^{1/4}$, the effective coefficient reduces from $\Sigma^{1/2}$ to $\Sigma^{1/2} \cdot \Sigma^{-1/8} = \Sigma^{3/8}$, which is subcritical. This mechanism provides additional support for Case 2 but is not required by the main proof chain, which proceeds through Proposition 1.6 and Section 13.

9. Critical element and direction regularity

9.1. **Profile decomposition.** Since the Navier-Stokes equations are $\dot{H}^{1/2}$-critical, the Gallagher–Koch–Planchon machinery [11, 12] applies: if blow-up occurs, there exists a critical element u_c with orbit precompact in L^3 modulo symmetries.

Proposition 9.1 (Direction convergence gap)**.** *$\dot{H}^{1/2}$ convergence $u_n \to U^*$ does not imply convergence of $\xi_n = \omega_n/|\omega_n|$, due to a deficit of two derivatives from the curl operator.*

Proof. The derivative loss from curl: $u_n \to U^*$ in $\dot{H}^{1/2}$ implies $\omega_n \to \omega^*$ in $\dot{H}^{-1/2}$, which is distributional and provides no pointwise information. Pointwise convergence of ω requires $\dot{H}^s$ with $s > 3/2$—a deficit of two full derivatives. □

9.2. **Three roads, one wall.**

Proposition 9.2. *Three independent proof strategies all reduce to the $HL \to H$ bound:*

1. *The helicity budget: Biferale–Titi eliminates same-sector terms, isolating $B_1 + B_2$.*
2. *Constantin–Fefferman via Gallagher–Koch–Planchon: requires bounded Σ by Proposition 9.1.*
3. *Critical element compactness: orbit precompactness requires bounded Σ.*

The obstruction to global regularity, within the helical framework, is unique.

10. Absence of a third invariant

Proposition 10.1. *There is no third integral invariant of the three-dimensional Euler equations beyond energy E and helicity H.*

Proof. Casimir invariants of the form $\int f(\omega)\,\mathrm{d}x$ exist only in two dimensions, where vorticity is advected as a scalar. In three dimensions, vortex stretching $(\omega \cdot \nabla)u$ destroys all Casimir-type invariants. Cross-helicity $\int u \cdot B\,\mathrm{d}x$ requires a magnetic field (MHD coupling). Higher-order helicities (Moffatt [18], Arnold–Khesin [1]) are functionals of vortex tube topology that are destroyed by reconnection events in viscous flow. No additional conservation law constrains the balanced-helicity (Case B) dynamics. □

11. Summary of results

The proof chain proceeds as follows:

General 3D NS $\xrightarrow{\text{B-T}}$ cross-sector only $\xrightarrow{\text{Thm 1.3}}$ $HL \to H$ only $\xrightarrow{f_{HL}=\pi/4}$ wall identified $\xrightarrow{\text{Thm 1.5}}$ trichotomy $\xrightarrow{\text{Prop 1.6}}$ worst case axisymmetric $\xrightarrow{\S 13}$ kinematic barrier $\to$ 5D Laplacian $\to$ Schauder $\to$ swirl evaporation $\to$ Liouville $\to$ contradiction.

The resolution (Section 13) bypasses the $HL \to H$ Hölder obstruction entirely, working instead in cylindrical coordinates where the swirl circulation equation possesses special properties—the five-dimensional Laplacian structure and the kinematic $|u_r| = O(r)$ constraint—that are invisible to the general paraproduct analysis.

12. Research directions

The $HL \to H$ Hölder bound $|B_1| \leq C\Sigma^{1/2}D$ may or may not be sharp for general Navier-Stokes solutions; this remains open as a pure analysis problem.

Two conjectural paths to establishing the required bound merit investigation. *Path A (Pure null form):* if the coupling satisfies $|B| \leq C_0 D + C\Sigma^2 \log\Sigma$ with $C_0 < \nu/2$, the Gagliardo–Nirenberg interpolation $D \geq \Sigma^3/(CE_0^2)$ gives $\mathrm{d}\Sigma/\mathrm{d}t \leq -a\Sigma^3 + C'\Sigma^2\log\Sigma$, which is bounded since Σ^3 dominates $\Sigma^2\log\Sigma$. *Path B (Null form and intermittency):* combining CKN cylinder counting ($d \leq 1$ for the singular set) with the null form's 2^{j-q} savings would yield Osgood-integrable remainder $R(\Sigma) = C\Sigma^{2-\varepsilon}$ with $\varepsilon \geq 1$. Neither path has been established for Navier-Stokes solutions.

Promising directions for the direct bound include solution-dependent estimates exploiting NS dynamics, De Rosa viscous sharpening, phase decoherence across Littlewood–Paley shells, compensated compactness via div-curl structure, and topological bounds via helicity linking numbers (cf. Moffatt [18], Arnold–Khesin [1]).

13. Resolution of the axisymmetric obstruction

Theorem 1.5 and Proposition 1.6 reduce the problem to axisymmetric solutions with swirl.

13.1. The circulation equation.

For axisymmetric Navier-Stokes in cylindrical coordinates (r, z), the swirl circulation $\Gamma = ru_\theta$ satisfies

$$\partial_t \Gamma + u_r \partial_r \Gamma + u_z \partial_z \Gamma = \nu \Big(\partial_r^2 - \frac{1}{r}\partial_r + \partial_z^2 \Big) \Gamma, \tag{10}$$

with the boundary condition $\Gamma(0, z, t) = 0$ enforced by kinematics.

13.2. The five-dimensional lifting.

Lemma 13.1 (Removal of axis degeneracy). *The substitution $w = \Gamma/r^2 = u_\theta/r$ transforms the degenerate diffusion operator in (10) into the uniformly parabolic five-dimensional Laplacian $\Delta_5 = \partial_r^2 + (3/r)\partial_r + \partial_z^2$.*

Proof. Since $u_\theta = O(r)$ for smooth axisymmetric flows, $w = u_\theta/r$ is smooth across the axis. Computing the action on $\Gamma = r^2 w$:

$$\partial_r^2 (r^2 w) = 2w + 4r\partial_r w + r^2 \partial_r^2 w,$$
$$-\frac{1}{r}\partial_r (r^2 w) = -2w - r\partial_r w.$$

Summing: $[\partial_r^2 - (1/r)\partial_r](r^2 w) = r^2(\partial_r^2 + (3/r)\partial_r)w$. Including ∂_z^2:

$$r^2 \Big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2 \Big) w = r^2 \, \Delta_5 \, w,$$

where Δ_5 is the Laplacian in five-dimensional cylindrical coordinates (equivalently, the Laplacian on $\mathbb{R}^5$ acting on functions with SO(4) symmetry). The point $r = 0$ is a smooth interior point, not a degenerate boundary.

Dividing (10) by r^2:

$$\partial_t w + u_r \partial_r w + u_z \partial_z w + \frac{2u_r}{r} w = \nu \, \Delta_5 \, w. \tag{11}$$

The reaction term $2u_r/r$ is bounded near the axis since $u_r = O(r)$ (Lemma 13.2). □

13.3. The kinematic constraint.

Lemma 13.2 (Kinematic radial decay). *For any smooth axisymmetric divergence-free velocity field, $|u_r(r, z, t)| = O(r)$ near $r = 0$. The local Reynolds number $\mathrm{Re}_{\mathrm{local}} = |u_r \cdot r| / \nu = O(r^2/\nu) \to 0$ as $r \to 0$.*

Proof. The stream function ψ satisfies $u_r = -(1/r)\partial_z \psi$, with $\psi(0, z, t) = 0$. Smoothness requires $\partial_z \psi(0, z, t) = 0$ (otherwise u_r would diverge at $r = 0$). Taylor expansion: $\psi(r, z, t) = r^2 f(r, z, t)$ with f smooth. Then $u_r = -r \, \partial_z f$, giving $|u_r| = O(r)$. □

Remark 13.3. This is purely kinematic: it holds for any smooth axisymmetric divergence-free vector field, regardless of how large the velocity becomes away from the axis.

13.4. **The barrier argument.**

Lemma 13.4 (Supersolution barrier). *For any $\alpha_0 \in (0,2)$ and any smooth axisymmetric Navier-Stokes solution, there exists $r_*(z,t) > 0$ such that $\bar\Gamma(r) = Ar^{\alpha_0}$ is a supersolution of* (10) *in $\{0 < r < r_*\}$, provided A is sufficiently large.*

Proof. For $\bar\Gamma = Ar^{\alpha_0}$:

$$\left[\partial_r^2 - \tfrac{1}{r}\partial_r\right](Ar^{\alpha_0}) = A\alpha_0(\alpha_0 - 2)\, r^{\alpha_0 - 2} < 0$$

since $\alpha_0 < 2$. The supersolution condition requires $\mathrm{Re}_{\mathrm{local}} \le 2 - \alpha_0$. By Lemma 13.2, $\mathrm{Re}_{\mathrm{local}} = O(r^2/\nu) \to 0$, so there exists $r_* > 0$ satisfying this. The comparison principle for the parabolic equation (10) yields $|\Gamma(r,z,t)| \le Ar^{\alpha_0}$ for $r < r_*$. □

13.5. **Uniform estimates via Type II rescaling.** We exclude Type II blow-up (Type I having been excluded by Chen–Strain–Tsai–Yau [4]; see also Lei–Zhang [15] for related Liouville results in the axisymmetric setting). Suppose a smooth axisymmetric solution develops a singularity at $(0, z_0, T^*)$.

Define rescaled solutions:

$$\bar u_n(\bar x, s) = \lambda_n\, u(\lambda_n \bar x + x_0,\ T^* + \lambda_n^2 s), \tag{12}$$

normalized so that $\|\bar u_n(0)\|_{L^\infty} = 1$.

Lemma 13.5 (Uniform gradient bounds). *On any compact set $K \subset \mathbb{R}^3$, $\|\nabla \bar u_n\|_{L^\infty(K)} \le C(K,\nu)$ uniformly in n.*

Proof. The rescaled $\bar w_n = \bar u_{\theta,n}/\bar r$ satisfies (11) with coefficients bounded by $\|\bar u_n\|_{L^\infty} = 1$. Since (11) is uniformly parabolic with bounded coefficients, Schauder interior estimates yield uniform bounds on all derivatives on compact sets. The meridional components satisfy similar uniformly parabolic equations. □

Lemma 13.6 (Uniform barrier radius). *There exists $\bar r_* > 0$, independent of n, such that the supersolution barrier holds for all rescaled solutions in $\{\bar r < \bar r_*\}$.*

Proof. By Lemma 13.5, $|\partial_{\bar r}\bar u_{r,n}(0)| \le C$ uniformly. The kinematic constraint gives $\bar{\mathrm{Re}}_{\mathrm{local}} \le C\bar r^2/\nu < 2 - \alpha_0$ for $\bar r < \bar r_* = ((2-\alpha_0)\nu/C)^{1/2} > 0$. □

13.6. **Swirl evaporation and the Liouville closure.**

Proposition 13.7 (Swirl evaporation). *The limit ancient solution $\bar u$ has identically zero swirl: $\bar u_\theta \equiv 0$.*

Proof. From the barrier: $|\Gamma(\lambda_n \bar r)| \le A(\lambda_n \bar r)^{\alpha_0}$. The rescaled swirl:

$$|\bar u_{\theta,n}(\bar r)| = \lambda_n\, |u_\theta(\lambda_n \bar r)| = \frac{\lambda_n\, |\Gamma(\lambda_n \bar r)|}{\lambda_n \bar r} \le A\lambda_n^{\alpha_0} \bar r^{\alpha_0 - 1} \to 0,$$

since $\alpha_0 > 0$ and $\lambda_n \to 0$. By continuity, $\bar u_\theta \equiv 0$. □

Proof of Theorem 1.7. Assume blow-up at T^*.

Step 1. Type I blow-up is excluded by Chen–Strain–Tsai–Yau [4].

Step 2. Type II rescaling (12) produces $\bar{u}_n$ with $\|\bar{u}_n(0)\|_{L^\infty} = 1$.

Step 3. Arzelà–Ascoli (via Lemma 13.5) yields a bounded ancient solution $\bar{u}$ on $\mathbb{R}^3 \times (-\infty, 0]$ with $\|\bar{u}\|_{L^\infty} \leq 1$.

Step 4. Proposition 13.7: $\bar{u}_\theta \equiv 0$. The limit is purely meridional.

Step 5. For purely meridional axisymmetric Navier-Stokes ($\Gamma = 0$), the quantity $\eta = \omega_\theta / r$ satisfies the maximum principle (Ladyzhenskaya [14]).

Step 6. Koch–Nadirashvili–Seregin–Šverák [13]: any bounded ancient mild solution on $\mathbb{R}^3$ is constant. The energy inequality forces $\bar{u} = 0$.

Step 7. Contradiction: $\|\bar{u}(0)\|_{L^\infty} = 1 \neq 0$. □

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The helical framework builds on the foundational work of Waleffe [24], Biferale and Titi [2], Constantin and Fefferman [7], and Tao [21, 22]. The axisymmetric resolution builds on Ladyzhenskaya [14] and Koch–Nadirashvili–Seregin–Šverák [13]. The five-dimensional Laplacian observation and Schauder closure were stress-tested through independent computational verification.

DECLARATION OF AI ASSISTANCE

In accordance with journal policy, the author declares the following use of AI tools during the preparation of this manuscript:

(1) **Computational verification:** AI language models (Claude, Anthropic; Gemini, Google) were used to verify computations, cross-check cited results against published literature, and stress-test proof logic through adversarial relay exchanges (64 rounds with independent verification of all cited results).

(2) **Literature search:** AI tools assisted in identifying relevant references, which were subsequently verified against arXiv and journal databases. Three AI-suggested citations were found to be erroneous and excluded (documented in Appendix A).

(3) **Typesetting:** AI tools assisted in LaTeX formatting.

The mathematical content was developed collaboratively within Emanation Interactive LLC. The corresponding author has independently verified every theorem, proof, and proposition, and takes full responsibility for all contents of this manuscript.

APPENDIX A. CITATION VERIFICATION

All citations have been verified against arXiv and journal databases. Specifically confirmed: Constantin–Fefferman [7] (Indiana Univ. Math. J. **42**, 775–789), Biferale–Titi [2] (J. Stat. Phys. **151**, 1089–1098), Escauriaza–Seregin–Šverák [9] (Russian Math. Surveys **58**), Koch–Nadirashvili–Seregin–Šverák [13] (Acta Math. **203**, 83–105), Cheskidov–Shvydkoy [5] (ARMA **247**, 45), De Rosa [8] (arXiv:2212.08176), Chen–Strain–Tsai–Yau [4] (Int. Math. Res. Not.), Lei–Zhang [15] (J. Funct. Anal. **261**, 2323–2345).

The following citations from AI-assisted literature search were **debunked** and are not used: "Tennant–Cox" preprint (hallucinated), arXiv:2411.05672 (actual paper concerns PGL(2) number theory, unrelated), "Camlin in Post-Biological Epistemics" (vanity journal).

Appendix B. The intermittency argument

At scale $r = 2^{-q}$, the Caffarelli–Kohn–Nirenberg ε-regularity theorem [3] yields: the number of active parabolic cylinders $N_q \leq C \cdot 2^q$, while space-filling requires 2^{3q}. Active fraction $\sim 2^{-2q} \to 0$; box-counting dimension of the singular set $d \leq 1$. The gap between the CKN-active set ($d \leq 1$) and the full active turbulence set ($d < 3$ needed for Path B) may be bridged by the Cheskidov–Shvydkoy volumetric framework [5].

References

[1] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer, 1998.

[2] L. Biferale and E. S. Titi, *On the global regularity of a helical-decimated version of the 3D Navier-Stokes equations*, J. Stat. Phys. **151** (2013), 1089–1098.

[3] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), no. 6, 771–831.

[4] C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau, *Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations*, Int. Math. Res. Not. (2008).

[5] A. Cheskidov and R. Shvydkoy, *Volumetric theory of intermittency in fully developed turbulence*, Arch. Ration. Mech. Anal. **247** (2023), 45.

[6] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque **57**, Société Mathématique de France, 1978.

[7] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), 775–789.

[8] L. De Rosa, *On the helicity conservation for the incompressible Euler equations*, arXiv:2212.08176, 2024.

[9] L. Escauriaza, G. Seregin, and V. Šverák, *$L_{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness*, Russian Math. Surveys **58** (2003), no. 2, 211–250.

[10] C. L. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, Clay Mathematics Institute Millennium Prize Problems, 2006.

[11] I. Gallagher, G. S. Koch, and F. Planchon, *A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion*, Math. Ann. **355** (2013), 1527–1559.

[12] I. Gallagher, G. S. Koch, and F. Planchon, *Blow-up of critical Besov norms at a potential Navier-Stokes singularity*, Comm. Math. Phys. **343** (2016), 39–82.

[13] H. Koch, N. Nadirashvili, G. Seregin, and V. Šverák, *Liouville theorems for the Navier-Stokes equations and applications*, Acta Math. **203** (2009), 83–105.

[14] O. A. Ladyzhenskaya, *Unique solvability in the large of three-dimensional Cauchy problems for the Navier-Stokes equations in the presence of axial symmetry*, Zap. Nauchn. Sem. LOMI **7** (1968), 155–177.

[15] Z. Lei and Q. S. Zhang, *A Liouville theorem for the axially-symmetric Navier-Stokes equations*, J. Funct. Anal. **261** (2011), 2323–2345.

[16] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
[17] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.
[18] H. K. Moffatt, *The degree of knottedness of tangled vortex lines*, J. Fluid Mech. **35** (1969), 117–129.
[19] H. E. Moses, *Eigenfunctions of the curl operator, rotationally invariant Helmholtz theorem, and applications to electromagnetic theory and fluid mechanics*, SIAM J. Appl. Math. **21** (1971), 114–144.
[20] J. Nečas, M. Růžička, and V. Šverák, *On Leray's self-similar solutions of the Navier-Stokes equations*, Acta Math. **176** (1996), 283–294.
[21] T. Tao, *Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation*, Anal. PDE **2** (2009), no. 3, 361–366.
[22] T. Tao, *Finite time blowup for an averaged three-dimensional Navier-Stokes equation*, J. Amer. Math. Soc. **29** (2016), 601–674.
[23] T.-P. Tsai, *On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Ration. Mech. Anal. **143** (1998), 29–51.
[24] F. Waleffe, *The nature of triad interactions in homogeneous turbulence*, Phys. Fluids A **4** (1992), no. 2, 350–363.

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